PERGAMON

International Journal of Heat and Mass Transfer 44 (2001) 3307–3320

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

A new finite integral transform pair for hyperbolic conduction problems in heterogeneous media

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Received 7 January 1999; received in revised form 17 July 2000

Abstract

The study of heat conduction in situations involving very short times cannot be set about with the help of Fourier's law. This model which cannot be justified in case of local non-equilibrium, leads, in this case, to results which do not agree with the observations. When the continuity hypothesis is posed, Fourier's law may nevertheless be adjusted. Among the different proposed models the Cattaneo–Vernotte model is one of the most credible. It is used in this work. The resulting hyperbolic problem in a non-homogeneous medium is posed in its primitive form of coupled system of partial differential equations where the unknowns are the temperature and the flux density, i.e. a four-component vector field. Thanks to a dot product which is suited to these fields, the existence of two orthogonal families of complex vector-eigenfunctions is shown. A finite-integral transform technique which is based on this dot product is then applied to the primitive system. It produces an infinite set of uncoupled ordinary differential equations: a separated expansion form of the solution vector field is thus obtained. For multifilmed media (1-D cases), it is shown that the notion of transfer matrix which is familiar in pure diffusion context generalizes naturally.

1. Introduction

Fourier's law which joins thermal flux density and temperature gradient linearly has proved its adequacy in numerous thermal engineering applications. Nevertheless, this law includes some defects and leads to paradoxical results in problems which involve high rates of temperature changes. So, Fourier's model implies an infinite heat propagation speed and, when applied to temperature-step problem, it leads to infinite thermal flux on boundaries. Moreover the model does not allow interpretation of experimental observations such as temperature waves in fluids or solids in cryogenic conditions [1–6] or in heterogeneous media at room temperature [7,8]. These observations justify giving up or, at least, putting right Fourier's law in certain circumstances. Similar defects appear in mass-diffusion for Fick's law [9]. These experimental and physical considerations require addressing the problem of what bounds to put to this standard constitutive law. Lower limits of time and space scales have been suggested [10] in homogeneous media; they are of the order of magnitude of the Boltzmann relaxation time for the time scale, and of the order of the mean free path of heat carriers for the space scale. The weakness of these limits explains why Fourier's law gives good results in the numerous circumstances where the physical observation scales are clearly higher than the above suggested scales. But the question of knowing what can be done in the vicinity or beneath the limits arises. This question includes two aspects. The first one is fundamental on a physical plane: how to carry out such studies? In particular, is it possible to modify Fourier's law in such a way as to correct the observed defects? The second side is pragmatic: is the engineer truly facing situations which force him to give up Fourier's law?

Thirty years ago the answer to the last question would have been no, he is not, because the evidence of defects was confined to rather exotic circumstances. But, today, the use of lasers which can deliver a high power

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0017-9310/01/$ - see front matter
PII: S0017-9310(00)00360-4

Nomenclature

$\mathbf{A}_i, \mathbf{B}_i$	matrix coefficients of the $\mathbf{L}[\cdot]$ operator
c	specific heat
C	studied domain boundary
D, D_i	studied domain, sub-domain
e	thickness
$\mathbf{L}[\cdot]$	partial differential operator
$\mathbf{N}_i$	kernel matrix of the dot-product
n	number of sub-domains
$\vec{n}_{ij}$	unit normal vector
$\hat{p}_k(t)$	transformed function
$\vec{\mathbf{Q}}(\vec{\mathbf{R}}),\ q(r)$	flux density in an eigenvalue problem (in adjoint problem), component
S	source-term
t	time
T	temperature
$\mathbf{T}$	vector field (thermal state vector)
U	first component of $\mathbf{Z}$
V	first component of $\mathbf{Z}^*$
W_α	given boundary condition
$\mathbf{Y}$	thermal resistance matrix
$y_{ij},\ y_i,\ y_\alpha$	thermal resistance
$\mathbf{Z}$	complex vector field

Greek symbols

β_i	$\beta_i = (\rho_i c_i \lambda_i / \rho_1 c_1 \lambda_1)^{1/2}$, local reduced effusivity
$\gamma_i(\omega, e_i)$	transfer matrix
φ	flux density (1-D problem)
$\vec{\Phi}$	flux density
$\vec{\Psi}$	flux density
λ	thermal conductivity
μ_i	$\mu_i^2 = \omega(1 - \tau_i \omega)$
ρ	density
τ	relaxation time
θ	vector field
ω_k	eigenvalue

Subscripts and superscripts

H	boundary homogeneous problem
I	initial value
i, j	sub-domain
k, l	eigenelement rank
S	pseudo-stationary
α	sub-domain external boundary
$\wedge$	finite integral transform
$*$	adjoint problem

during very short time intervals has become a current practice in several engineering applications such that the scale bounds may be easily reached. Vedavarz et al. [11] have analyzed these practical bearings. They conclude that non-Fourier effects may have significance in pulse-laser heat treatment of metals or semiconductors, the same is true in laser surgery or else in thin film applications for which Qiu and Tien [12] specifically quote diamond film treatment as Jen and Chieng [13] do. Chen [14] adds other engineering applications in the field of nanofabrications. All papers are consistent with the part of non-Fourier heat conduction in numerous new engineering technologies [15] which are neither limited to space nor to cryogenic multilayer applications [16].

There are two main ways of studying heat conduction at microscale.

The first way consists in a return to elementary mechanisms by applying the laws of mechanics at a microscopic level. This means is used in molecular dynamics simulations. Thus a purely mechanical approach gives access to the energy carried by atoms. Temperature and flux densities at a macroscopic level may then be calculated as soon as time and space scales give a meaning to these variables. In a crystalline medium, this technique leads to a limit space scale which is less than the inter-atomic distance although the limit time scale is of the order of the oscillating period of the atom [17,18]. Both scales are much lower than the standard thermodynamics limits. Thus molecular dynamics gives access to the thermal properties of some materials [17,19]. Moreover, transient modeling may be used to check the validity of macroscopic models [17,18] (we will return to that point later). However molecular dynamics calculations present some constraints. On the one hand a realistic model of the interaction potential must be known, which is not always the case as Hipwell and Tien [20] noticed and, when it is known, the model must be fitted to operating conditions [19]. On the other hand, powerful calculation means and efficient numerical schemes are required since the considered number of atoms is very large. Most of the time, only a periodic medium may be studied. Amorphous media seem to be excluded [21].

The second way is more classical in thermal engineering. It relies on continuity hypothesis. As soon as this hypothesis is posed, the energy balance and the internal constitutive law are expressed in differential form. The latter law often has a purely phenomenological origin but may be reinforced by intermediary notions and theories (thermodynamics, quanta, etc.). This way is more familiar to engineers and puts the research of the solution of the problem on the well-known ground of partial differential equations. It allows to consider realistic space domains into a single continuous framework. The drawback of this second way is that it implies two hypothesis steps (a first step for continuity and a

second one for the constitutive law), thus in case of deficiency, the origin of the malfunction is not easy to delimit: is it the continuity hypothesis? is it the flux law? is it, even, a wrong solution of the mathematical problem?

The aim of this paper compels us to review continuous macroscopic models in a more detailed way.

2. Continuous macroscopic models

The most pertinent models rely on the idea of nonlocality, in time, of the energy transfer. It was proposed in [22] and was then generalized in [23]. The phenomena are governed by a system of three equations which express:

- firstly, the energy balance which links the internal energy rate to the flux-density divergence;
- secondly, the definition of the internal energy;
- thirdly, the definition of the flux density.

The last two equations are integro-differential equations which integrate memory effects through a kernel function. Assuming the standard equilibrium definition of the internal energy, this last state variable may be eliminated between the first two equations and, by considering then an exponential heat-flux memory kernel, the following two equations (1-D case) are obtained:

$$\rho c \frac{\partial T}{\partial t} + \frac{\partial \varphi}{\partial x} = 0, \tag{A}$$

$$\tau_\varphi \frac{\partial \varphi}{\partial t} + \lambda \frac{\partial T}{\partial x} + \varphi = 0. \tag{B}$$

Eq. (B) is the constitutive equation which was originally proposed by Cattaneo [24] and Vernotte [25]. With less standard definitions of the internal energy, it is possible to obtain other kinds of models. Among them, the model, which leads to the modified hyperbolic equation [26,27] in which a flux-density quadratic term is added to the equilibrium internal energy, may be mentioned. In the same way, by using a concept of thermal inertia, a second relaxation constant τ_T, which controls the internal energy may be introduced. The dual-phase lag model of Joseph and Preziosi [28] and Tzou [15,29] is then recovered. In accordance with the interpretation of the last author, the delayed response between the flux-density and the temperature gradient (τ_φ) and between the temperature gradient and the temperature (the so-called thermalization constant, τ_T) finds its origin in microscopic mechanisms. Honner and Kunes [30] recently proposed a simplified classification of this dual-phase lag model which could describe the thermal behavior of a great variety of homogeneous or heterogeneous media. According to [30], when the ratio τ_T/τ_φ is less than one unit, the model captures a dominant mechanism of elastic collision of phonons and the thermal behavior is wavy, as the ratio approaches the unit, the inelastic collisions gain weight, which leads to a more and more diffusive behavior. A ratio greater than the unit as in Joseph and Preziosi's model [28] would correspond to heterogeneous solids. In that case the model would capture the interaction between different solid phases acting as heat carriers. The distinction between micro- and macro-scales is then a question of agreement only: the observation scale is the macroscopic scale, smaller scales correspond to heterogeneities and are microscopic scales. This parallel conduction ($\tau_T/\tau_\varphi > 1$) interpretation [30] agrees with the anomalous diffusion which was revealed by Hipwel and Tien [20] as a consequence of the interactions between different phases in a random purely diffusive medium when the time scale of excitation interacts with the diffusion time scale of the random medium. No thermal wave can occur in this case. A similar local scheme, i.e. the parabolic two-step model has been proposed by Qiu and Tien [12] to describe the thermal behavior of metals in short-pulse laser heating. The electrons and the lattice appear as two "phases" in local interaction and when the electron temperature is eliminated between the two equations which govern the coupled thermal behavior of both "phases", the dual-phase lag model is recovered [31]. In the same way, the hyperbolic two-step model [32] reduces to the dual-phase lag model if second-order effects (τ_φ^2 and τ_T^2 terms) are missed (see [15]). The previous analysis shows that two main continuous models prevail: the Cattaneo–Vernotte model (C–V) and the dual-phase lag (DPL) model. Owing to their phenomenological origin, they must be compared with different tests. This is what we shall examine now.

Irreversible thermodynamics allows to justify Fourier's law which appears to be the simplest linear form compatible with the second law. When the medium is submitted to high temperature gradients, the local equilibrium hypothesis, which is used for recovering Fourier's law, fails. The new frame of Extended Irreversible Thermodynamics [33], thanks to a new definition of the entropy which now depends on the flux, allows a complementary term to appear in Fourier's law. This term is similar to the τ_φ's term in Eq. (B) [34]. This test thus confirms the C–V model. Literature does not show DPL model recovering through the extended irreversible thermodynamics theory. However, another thermodynamic frame allows to obtain a model with two relaxation constants [35] but, firstly the theory postulates Fourier's law, i.e. local equilibrium, and secondly, in the obtained equations the time constants appear in inverted positions in comparison with the DPL model (as if τ_φ was the thermalization constant and vice versa).

Another kind of test may be found in the microscopic foundation of continuous laws. Physical deficiencies of Fourier's law result from its inability to take account of the microscopic structure of materials [36]. Thanks to

statistical methods, the thermodynamic state of a system which is formed by identical particles may be defined. Volz [17] thus showed that the C–V law may be recovered in weak non-equilibrium regimes. The same author goes forward when doing direct transient molecular dynamics simulations. The solution of the motion equations of a set of interacting atoms permits him to determine, by stochastic mean calculations, temperature and flux density at a given time for the considered set or for subsets [17,18]. The simulation concerns an argon crystal. The results are compared to the hyperbolic equation results (C–V model) and to the modified hyperbolic equation [26,27]. There is no agreement with the latter model and a partial agreement with the C–V model. The Clausius experiment gives a rather correct agreement [17,18] but the given temperature wall case shows a clear disagreement, notably for the time behavior of the flux [17].

A third possible approach for determining the thermal energy transfer at microscale is based on phonon dynamics. Since phonons are particles, their density is governed by the Boltzmann transport equation, Tavernier [37] showed that, by modeling the collision term of the Boltzmann transport equation with a time relaxation constant, the flux density law is similar to the C–V model. In the same way, Qiu an Tien [32] used the BTE for describing the electron energy transport in metals and thus justified the hyperbolic two-step model which, as explained above, comes down to the DPL model.

All these justifications do give some consistence to the C–V model but the DPL model is not dismissed. Nevertheless the proofs remain rather theoretical proofs. They show a consistency between different facets of physics but they are not final proofs because they need complementary hypothesis inside the course of demonstrations. Since the models are, of course, intended to estimate phenomena which may be truly observed, the most convincing proof is the experimental proof.

Regarding homogeneous media the main experimental difficulty is the weakness of space and mostly time scales which must be reached to bring to the fore non-Fourier effects. As a matter of fact, in the phonon dynamics approach, the C–V time constant τ_φ appears to be of the same order as the Boltzmann relaxation constant [17,18], i.e., few picoseconds in cryogenic conditions, for setting an order of magnitude. It indeed depends on the state of aggregation (it is weaker in solids than in gas or plasma where it reaches several nanoseconds [30] but it decreases as temperature increases (see [11])). The thermalization time τ_T of the DPL model is also very small but, in metals, it might become a hundred times higher than the flux relaxation constant [15]. In heterogeneous media the parallel conduction approach [30] shows that in solid-phase media τ_φ may reach the nanosecond in composites (and τ_T is always higher than τ_φ). The general weakness of characteristic time scales led experimenters to work in the most comfortable conditions regarding the response time of measuring devices. This is the reason why most results concern non-Fourier effects either under cryogenic conditions or in heterogeneous media.

The existence of temperature waves was proved as early as in 1944 [38] in liquid Helium. Since then, numerous authors have observed this phenomenon and measured wave speeds [1–5]. These observations go along the same line as the C–V model (or DPL model when $\tau_T/\tau_\varphi \ll 1$) and, recently again, Torczinski [6] underscores the interface partial reflection of waves in a two-phase medium. No convincing result of this kind seems to exist at room temperature in homogeneous media. Nevertheless, Qiu and Tien experiments [12] on gold film at room temperature agree with the parabolic two-step model which indirectly validates the DPL model when $\tau_T/\tau_\varphi > 1$. Other experiments at room temperature concern heterogeneous media. Kaminski [7] reports τ_φ values up to 10 s in various porous media, though Mitra et al. [8] measure, in processed meat, τ_φ values near 15–16 s. These last results validate the C–V model in a Clausius-kind experiment but, under a given temperature boundary condition some irreducible discrepancies remain [8]. We notice that these results do not agree with the simplified analysis of [30] for parallel conduction (in solids) but, it is true that processed meat is far from being a solid mix at nearly room temperature which is considered by the authors.

At this stage of the bibliography study we may draw some first conclusions.

The existence of non-Fourier heat conduction is undoubted. The associated phenomena appear when the thermal exciting input which is created either inside the medium or at the boundaries has a weak time scale when compared to proper scales of heat carriers. Cattaneo–Vernotte model is the most credible continuous model owing to, both, its theoretical support and the experimental results which show a wave-like heat transport. The dual-phase lag model is interesting in view of its easy adaptation to a large phenomenological range which goes from quasi-hyperbolic conduction ($\tau_T/\tau_\varphi \ll 1$) to parallel conduction ($\tau_T/\tau_\varphi \gg 1$) undergoing intermediary regimes of anomalous diffusion.

Nevertheless not everything is perfectly clear. In particular, it has been observed that under some initial and/or boundary conditions both models give unsatisfactory results. For instance some initial conditions generate an overshooting which characterizes by internal temperatures which are higher than the initial and boundary temperatures [31] at the same time. The same overshooting phenomenon was emphasized by Bai and Lavine [39] and also Kronberg et al. [40] who call the boundary conditions in question.

Both models require two initial conditions. When, as in a majority of studies, the differential problem is posed

after the flux density is eliminated, the temperature field and the time-derivative temperature field must be given. On a mathematical basis, a great variety of fields may be given, but on a physical basis the given fields must be observable fields. Thus, very often in the literature concerning the hyperbolic heat equation, equilibrium fields – which are observed by maintaining medium boundaries at the same constant temperature for a long time – are given. A typical transient study consists then in obtaining the temperature change inside the medium under new boundary conditions. The response to step-functions has been widely studied probably because it leads to spectacular results. In this particular case, we should point out the impossibility of abruptly varying (at time $t = 0$), say, the temperature of a boundary but, at the same time, maintaining its time-derivative to a zero value; the response will surely retain marks of this impossibility. Nevertheless, few authors actually considered this aspect of the problem when studying the response to step-functions and few saw the necessity of using Dirac's functions [41,42] in these circumstances. This is nothing but an example, but, in a general way improper initial conditions might explain some unphysical results.

On the other hand, C–V law takes into account the existence of local non-equilibrium inside the medium, but, a non-equilibrium situation also exists at all interfaces, i.e. at inside interfaces in composite-like media and at external interfaces (boundaries). This kind of non-equilibrium has, at first, its origin in space effects, therefore it exists in stationary regimes as much as in transient regimes. Using the BTE, Chen [14] showed that in steady regime the emitted flux from a particle which is embedded in a host medium was less than the Fourier-predicted flux, as soon as the particle dimension was less than the heat carrier mean free path. In a similar way Jen and Chieng [13] showed that the grain structure of media or of interfaces between two media modifies the heat transfer rate. A temperature gap appears at the interface between the studied diamond film and the silicon substrate; the corresponding thermal resistance depends both on the structure and on the dimension of grains near the interface. With the help of a different methodology, Kronberg et al. [40] obtain a similar conclusion for boundary conditions and show that Dirichlet conditions are not compatible with the C–V model. A third-kind boundary condition must be used and this condition makes the overshooting anomaly disappear. Thus, just like Fourier's constitutive law must be modified when a medium is subjected to a strong internal non-equilibrium, a specific constitutive law must be used at interfaces. A standard thermal resistance model where the thermal resistance value results from microscopic calculations may be used. Of course, when two identical media are put into contact, the thermal resistance vanishes [40]. Inside a non-homogeneous medium, such as a composite medium for instance, the interface constitutive law is complemented by the energy balance equation. For the C–V model, owing to the fact that the outgoing energy from a given medium immediately enters the adjacent medium, the energy balance is expressed by the continuity of the flux density. It would not necessarily be the same in other contexts. For instance Bai and Lavine [39] suggested a flux density gap at interfaces as a result of the flux-dependent internal energy. It would be the same in models which include a coupling between thermal and mechanical phenomena. Flux-density preservation at internal interfaces does not foresee anything about partial reflections. The results of Frankel et al. [41] in their study of a two-layered wall clearly show that such reflections exist at the interface between two different media under a continuity condition of both flux and temperature.

As soon as the model is chosen and the initial and boundary conditions are well (or not well) posed, its use for comparing its consequences to experimental or other results, goes through the solution of a partial differential problem.

A majority of works focus on linear problems. In two-dimensional cases [43–45] Green's function method [43] or Laplace transform are used. They are associated with numerical techniques either in the inversion problem [43] or in the solution of the space problem [45]. The 1-D studies in homogeneous media are the most numerous. For the C–V model, the considered equation is almost always the hyperbolic heat conduction equation where the unknown is either the temperature or the flux density. Standard analytical methods such as Laplace transform [46,47], Green's function [42,48] or finite-integral transform [11] are available. When boundary conditions concern both temperature and flux density, the elimination of one of the state variables (either the flux or the temperature) which is necessary for obtaining a well-posed problem in the hyperbolic heat conduction equation context, poses the problem of admitting that the internal constitutive law is also verified at boundaries. Moreover, we have explained above that generalized functions must be used when studying the response to step-functions. An elegant way which suppresses these difficulties consists in keeping both state variables in the formulation. It is what is generally done in purely numerical solutions (see [17] or [49] for examples) but the same way may be followed in a solution through Laplace's method [40]. Standard, or less standard analytical methods, as the method of characteristics [50], are always available for 1-D linear problems, nevertheless, the – at least partial – resort to numerical techniques may turn out to be useful. Thus, Vedavarz et al. [11] sometimes used a numerical integration scheme of the method of characteristics, Qiu and Tien [12] implemented a Crank–Nikolson scheme in their study of laser heating of metals, Tzou and Chiu [51] make use of

the Kutta–Runge method for time-integrating. Tzou et al. [31], as for them, use a numerical inversion of the Laplace transformed problem.

Macroscopically non-homogeneous media make particular difficulties appear. Very few works are devoted to this subject. Frankel et al. [41] studied the thermal response of an adiabatic two-layered wall which is excited by a time evanescent internal source. They considered the Cattaneo-Vernotte model in the scope of a flux formulation of the hyperbolic heat conduction equation. Thus, the interface flux density continuity condition is easily written but the temperature continuity condition (only perfect thermal contact is considered) makes a new kind of interface time-dependent condition appear. The solution of the problem is searched by using a finite integral transform which is fitted to the wave equation. This transform does not separate the variables: an infinite set of time differential equations which are coupled to each other appear in the transformed space. This work shows clearly the present deficiency of finite integral transform methods in non-Fourier context. The full separation of the space and time variables appears only in media where the relaxation constant τ_φ is the same all over the medium (see [41]).

As a second conclusion of this part of the bibliography study we emphasize the following points.

Some defects of linear model in non-Fourier context might be linked to the use of improper initial and/or boundary conditions. Among the methods of solution of linear models, it seems better to accept methods which retain both physical state variables (temperature and flux) because this formulation calls for less hypothesis and less elaborate mathematical tools (for response to step-function studies). The finite integral transforms which have been used until now in hyperbolic heat conduction are really well fitted to homogeneous media only because they do not separate the variables in non-homogeneous media.

In a general way the literature shows that non-homogenous media have not been studied much in spite of their technological significance in thin film applications. It is then justified to take an interest in these media. First two points are to be considered for studying the non-Fourier conduction in that case. What is the chosen internal constitutive law? what is the interface constitutive law? As soon as these choices are made, the subsidiary question is: how to formulate the initial and boundary conditions so as to ensure a physical and mathematical consistency. Lastly, a method of solution which fits the resulting mathematical problem must be found. The relevance of the answers to these questions must then be given through different validation steps concerning the solution method, the internal model, the interface model, etc.

In this work, we are going to choose the Cattaneo–Vernotte model as internal law and a standard thermal resistance model as interface constitutive law. In this first article we limit our objective to set up a new tool of solution which fits into the non-homogeneous media. In a subsequent step we will present a validation of the method and then we will use it in the view of justifying the internal model choice.

The interest of finite-integral transform methods as it appears in purely diffusion context [52–56] is in the easiness of inversion technique. Thanks to the separation of the variables the solution may be written in the form of an expansion of the temperature field on a space eigenfunction basis where each time-dependent coefficient of the development is calculated independently of other coefficients. This is true in linear conduction problems, this is also true in some convection and linearized radiation problems [57] provided the proper eigenfunctions of the space operator are used. In full non-linear problems [58] this is not true, of course, because of the lack of a proper eigenbasis. The following work is built around the use of the proper eigenbasis of the linear partial differential operator which governs both physical state variables, temperature and flux density in non-Fourier conduction. In a first part it is showed how to obtain the proper eigenfunctions in a general 3-D non-homogeneous medium and what is the new finite integral transform (FIT) which separates the variables. In a second part we consider the special case of multilayered walls (1-D problems) and we show that the known quadrupole methodology remains valid in this linear non-Fourier context.

3. The 3-D finite integral transform

3.1. Basic governing equations

The studied domain (D) is an assembly of n adjoining passive sub-domain (D_i) (Fig. 1). Though it is not a true limitation, each sub-domain has constant thermal properties. Inside each sub-domain, the local energy balance involves

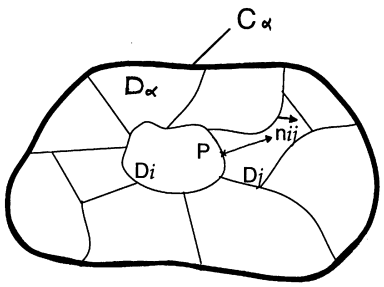

Fig. 1. Sketch of the studied domain.

$$\frac{\partial T_i}{\partial t} + \frac{1}{\rho_i c_i}\vec{\nabla}\cdot\vec{\Phi}_i = 0, \quad M_i \in D_i \tag{1a}$$

and the constitutive law writes

$$\frac{\partial \vec{\Phi}_i}{\partial t} + \frac{\lambda_i}{\tau_i}\vec{\nabla}\cdot T_i + \frac{\vec{\Phi}_i}{\tau_i} = 0, \quad M_i \in D_i. \tag{1b}$$

These two equations are complemented by two connecting laws

$$\vec{\Phi}_i \cdot \vec{n}_{ij} + \vec{\Phi}_j \cdot \vec{n}_{ji} = 0, \quad P \in C_{ij}, \tag{2a}$$

$$T_i - T_j = y_{ij}\vec{\Phi}_i \cdot \vec{n}_{ij}, \quad P \in C_{ij}. \tag{2b}$$

Eq. (2a) expresses the energy balance on the boundary (C_{ij}) which is common to the adjacent sub-domains (D_i) and (D_j), while Eq. (2b) is the contact constitutive law which includes the thermal resistance y_{ij}.

On the boundary (C) of the domain (D) the limiting conditions are

$$T_\alpha - y_\alpha \vec{\Phi}_\alpha \cdot \vec{n}_{\alpha 0} = W_\alpha(P,t), \quad P \in C_\alpha \tag{3}$$

knowing that all the C_α-boundaries constitute the external boundary (C) of the studied domain. The Eqs. (1a)–(3) are verified for positive time; at the initial time, the temperature and the flux density fields are given in the whole domain:

$$T_i = T_i^{\mathrm{I}}(M_i), \quad M_i \in D_i \cup C_i, \quad i = 1,n, \ \ t = 0, \tag{4a}$$

$$\vec{\Phi}_i = \vec{\Phi}_i^{\mathrm{I}}(M_i), \quad M_i \in D_i \cup C_i, \quad i = 1,n, \ \ t = 0. \tag{4b}$$

We observe, at this stage, that, by defining the vector field

$$\mathbf{T} = \{\mathbf{T}_i, \ i = 1,n\}, \ \text{where } \mathbf{T}_i^t = (T_i, \Phi_{i1}, \Phi_{i2}, \Phi_{i3})$$

the system of equations (1a) and (1b) writes

$$\frac{\partial \mathbf{T}}{\partial t} + \mathbf{L}[\mathbf{T}] = 0,$$

where $\mathbf{L}[\cdot]$ is a linear partial differential operator. This operator may be explicited, either in matrix form or in a semi-analytical form. This latter form will rather be used in this first part. It writes

$$\mathbf{L}[\mathbf{T}] \equiv \left\{ \begin{array}{l} \dfrac{1}{\rho_i c_i}\vec{\nabla}\cdot\vec{\Phi}_i \\ \dfrac{\lambda_i}{\tau_i}\vec{\nabla}T_i + \dfrac{\vec{\Phi}_i}{\tau_i}, \quad i = 1,n \end{array} \right\}.$$

3.2. Diagonalization of the operator. Eigenfunctions

Let $\mathbf{Z}$ and $\mathbf{Z}^*$ be two four-components complex vector fields. Each field is continuous in each sub-domain and has sufficient derivability properties with regard to the space variables. The dot-product of the two fields is defined by:

$$\langle \mathbf{Z}, \mathbf{Z}^* \rangle = \sum_{i=1}^{n} \int_{D_i} \mathbf{Z}_i^t \mathbf{N}_i \bar{\mathbf{Z}}_i^* \, \mathrm{d}v_i,$$

where

$$\mathbf{N}_i = \mathrm{diag}\left(\rho_i c_i, \frac{\tau_i}{\lambda_i}, \frac{\tau_i}{\lambda_i}, \frac{\tau_i}{\lambda_i}\right).$$

The semi-analytical expression of this dot-product is

$$\langle \mathbf{Z}, \mathbf{Z}^* \rangle = \sum_{i=1}^{n} \int_{D_i} \left(\rho_i c_i U_i \bar{V}_i + \frac{\tau_i}{\lambda_i}\vec{\mathbf{Q}}_i \cdot \vec{\bar{\mathbf{R}}}_i\right) \mathrm{d}v_i,$$

where U_i (resp. V_i) is the first component of $\mathbf{Z}_i$ (resp. $\mathbf{Z}_i^*$) and $\vec{\mathbf{Q}}_i\cdot$ (resp. $\vec{\bar{\mathbf{R}}}_i$) is a vector which is built with the three remaining components of $\mathbf{Z}$ (resp. $\mathbf{Z}^*$). The $\mathbf{Z}$ field is subject to the interfacial conditions (2a) and (2b) and to the boundary conditions of the homogeneous problem (which are obtained by setting to zero the right-hand side of Eq. (3)), whereas the $\mathbf{Z}^*$ field is subject to the homologous conditions of the adjoint problem.

The adjoint operator $\mathbf{L}^*[\cdot]$ is defined by (Appendix A)

$$\mathbf{L}^*[\mathbf{Z}^*] = \left\{ \begin{array}{l} -\dfrac{1}{\rho_i c_i}\vec{\nabla}\cdot\vec{\mathbf{R}}_i \\ -\dfrac{\lambda_i}{\tau_i}\vec{\nabla}V_i + \dfrac{\vec{\mathbf{R}}_i}{\tau_i}, \quad i = 1,n \end{array} \right\}$$

with the associated homogeneous conditions

$$\vec{\mathbf{R}}_i \cdot \vec{\mathbf{n}}_{ij} + \vec{\mathbf{R}}_j \cdot \vec{\mathbf{n}}_{ji} = 0,$$
$$V_i - V_j = -y_{ij}\vec{\mathbf{R}}_i \cdot \vec{\mathbf{n}}_{ij}.$$

Thus, by diagonalizing both operators, that is to say, by finding the solutions $\{\omega_k, \ \mathbf{Z}^k\}$ and $\{\omega_k^*, \ \mathbf{Z}^{*k}\}$ of the eigenvalue problems

$$\mathbf{L}\left[\mathbf{Z}^k\right] = \omega_k \mathbf{Z}^k \tag{5a}$$

and

$$\mathbf{L}^*\left[\mathbf{Z}^{*l}\right] = \omega_k^* \mathbf{Z}^{*l}, \tag{5b}$$

where, of course, $\mathbf{Z}^k$ and $\{\mathbf{Z}^{*l}\}$ verify the interfacial conditions and the homogeneous boundary conditions, we obtain two eigensets. Some properties of the eigenelements are given in Appendix B where, in particular, it is shown that $\mathbf{Z}^k$ is orthogonal to $\mathbf{Z}^{*l}$ and that, because of the links between $\mathbf{Z}$ and $\mathbf{Z}^*$, only one of the two problems (5a) and (5b) is to be solved.

Since the eigenvector field $\{\mathbf{Z}^k\}$ forms a basis for any vector field, $\mathbf{T}^{\mathrm{H}}$, which verifies, the same general mathematical properties as $\mathbf{Z}$, the interfacial conditions and the homogeneous boundary conditions, this field may expand on this eigenbasis

$$\mathbf{T}^{\mathrm{H}} = \sum_k \hat{p}_k^{\mathrm{H}} \mathbf{Z}^k. \tag{6}$$

The following section shows how to obtain the $\hat{p}_k^{\mathrm{H}}$ projections and how the solution of the given problem (1a)–(4b) proceeds.

3.3. The finite integral transform and its use

The linearity of the posed problem permits us to look for its solution by superposing two fields

$$\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{H}}.$$

The field $\mathbf{T}^{\mathrm{S}} = \{\mathbf{T}_i^{\mathrm{S}}(M_i,t),\ i=1,n\}$ is a quasi-stationary field which obeys

$$\left\{ \begin{array}{ll} \dfrac{1}{\rho_i c_i}\vec{\nabla}\cdot\vec{\Phi}_i^{\mathrm{S}} = 0, & M_i \in D_i \\ \lambda_i \vec{\nabla} T_i^{\mathrm{S}} + \vec{\Phi}_i^{\mathrm{S}} = 0, & M_i \in D_i \end{array} \right\}, \quad i = 1,n.$$

$\mathbf{T}^{\mathrm{S}}$ verifies also the interfacial conditions (2a) and (2b) and the full boundary conditions (3). The time is a dummy parameter in this problem which corresponds to the standard heat conduction problem in stationary regime. We suppose that its solution is known.

The field $\mathbf{T}^{\mathrm{H}} = \{\mathbf{T}_i^{\mathrm{H}}(M_i,t),\ i=1,n\}$ obeys

$$\frac{\partial \mathbf{T}^{\mathrm{H}}}{\partial t} + \mathbf{L}\left[\mathbf{T}^{\mathrm{H}}\right] + \mathbf{S} = 0, \tag{7}$$

where the source-term $\mathbf{S}$ is the time-derivative of the pseudo-stationary field $\mathbf{T}^{\mathrm{S}}$. The field $\mathbf{T}^{\mathrm{H}}$ verifies the interfacial conditions and the homogeneous boundary conditions; its initial value is simply the difference between the initial value of the full field $\mathbf{T}^{\mathrm{I}}$ and the value of the quasi-stationary field when the dummy parameter t is 0

$$\mathbf{T}^{\mathrm{H}}(M,0) = \mathbf{T}^{\mathrm{I}}(M) - \mathbf{T}^{\mathrm{S}},(M,0), \quad M \in (D),\ \ t = 0.$$

Let $\hat{p}_k(t)$ be the finite integral transform of any field Θ which is defined in the (D) domain. We define this FIT as the projection of Θ on the vector field $\mathbf{Z}^{*k}/\langle \mathbf{Z}^k, \mathbf{Z}^{*k}\rangle$

$$\hat{p}_k(t) = \frac{1}{\langle \mathbf{Z}^k, \mathbf{Z}^{*k}\rangle} \sum_{1=1}^{n} \int_{D_i} \left(\rho_i c_i \theta_i \bar{\nabla}_i^k + \frac{\tau_i}{\lambda_i} \vec{\Psi}_i \cdot \vec{\mathbf{R}}_i^k \right) \mathrm{d}v_i. \tag{8}$$

When applied to the Eq. (7), the finite integral transform (8) leads to the initial-valued problem

$$\frac{\mathrm{d}\hat{p}_k^{\mathrm{H}}}{\mathrm{d}t} + \omega_k \hat{p}_k^{\mathrm{H}} + \hat{s}_k = 0, \tag{9a}$$

$$\hat{p}_k^{\mathrm{H}}(0) = \frac{\langle \left(\mathbf{T}^{\mathrm{I}}(M) - \mathbf{T}^{\mathrm{S}}(M,0), \mathbf{Z}^{*k}\right)\rangle}{\langle \mathbf{Z}^k, \mathbf{Z}^{*k}\rangle}. \tag{9b}$$

Some properties of the $\mathbf{L}^*[\cdot]$ operator on the one hand, and the links between the eigenvalues ω_k and ω_k^* on the other hand, are used for obtaining the second term of the Eq. (9a). The solution of the problem (9a) and (9b) writes

$$\hat{p}_k^{\mathrm{H}}(t) = \hat{p}_k^{\mathrm{H}}(0)\,\mathrm{e}^{-\omega_k t} - \int_0^t \hat{s}_k(s)\,\mathrm{e}^{-\omega_k (t-s)}\,\mathrm{d}s. \tag{10}$$

Since the vector field $\mathbf{T}^{\mathrm{H}}$ obeys the conditions which allow to expand it on the eigenbasis $\mathbf{Z}^k$, property (6) is valid. Eq. (6) is thus the inverse transform of (8); as might be expected, the inverse transform is particularly straightforward.

The solution of the problem which is defined by the Eqs. (1a)–(4b) may thus be written

$$\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \sum_k \hat{p}_k^{\mathrm{H}}(t)\mathbf{Z}^k.$$

3.4. First comments

Throughout the above exposition, the differential problem is always considered as turning on an unknown vector field $\mathbf{T}$; the proposed method of solution of the hyperbolic heat conduction in heterogeneous medium uses the generalization of the finite integral transform method to function vector fields. Thus, both fundamental physical informations on heat transfer, i.e. the temperature and the flux densities continuously appear in the formulation. As already explained in Section 2, this is an advantage on both physical and mathematical points of view. We notice that, when building the solution the same time-coefficient (the integral transform, $\hat{p}_k^{\mathrm{H}}(t)$) controls the temperature and the flux density series expansions; the counterpart is that both fields must, of course, be known at the initial time.

In the same spirit, the use of a dot product which fits function vector fields leads easily to the true separation of the variables: each time coefficient, $\hat{p}_k^{\mathrm{H}}(t)$ is independent of other $(\hat{p}_l^{\mathrm{H}}(t))$ coefficients; but, this separability property is obtained at the cost of an intricacy of the eigenvalue problem since the eigenelements are complex-valued elements. As a matter of fact, since the considered problem is not a self-adjoint problem – a fundamental difference with the pure thermal diffusion case – the solution will gain its wave propagating behavior through the complex eigenvalues.

4. One-dimensional problem

When each sub-domain (D_i) has a slender look, thermal transfers across the (D) domain accept a 1-D modeling, provided that the boundary and initial conditions allow so.

We now place ourselves in this restricted scope; the medium is thus formed by the juxtaposition of layers. The non-Fourier modeling may be justified through the small value of the layer thicknesses: as a layer thickness diminishes, the time diffusion scale of this layer reduces and gets closer to the Vernotte relaxation time. But there

are limits to the reduction of time diffusion scale through the space scale since the continuity hypothesis may fall down. The literature puts this limit slightly beneath the mean free path of heat carriers (see [17, p. 96]). Therefore this limit depends on the properties of the medium itself and of the temperature; when the layer is a true homogeneous medium – as opposed to porous media which are seen as homogeneous media at the observation scale but are heterogeneous at lesser scales – the limit ranges from a fraction of nanometer to few nanometers.

In this case the studied wall may be seen as a multifilmed coating where, as in the first part, interface phenomena are taken into account with the help of a thermal resistance.

4.1. Transfer matrices for hyperbolic conduction

The theoretical justifications given in the first part of this work remain, of course, valid for 1-D problems; but since in this case, there remains only one space variable in the problem, the diagonalization of the operator will lead to an ordinary differential problem. We should be able to find an analytical solution of the eigenvalue problem. We already know that, in pure conduction context, the $\mathbf{T}_i$ vector field which has now two components T_i and φ_i, permits us to rationally introduce the transfer matrices [59] which are a rather easy calculation means for studying the heat transfer in multilayered walls [55,56,59–61]. We show below that the same is true for hyperbolic conduction.

By using the local reduced variables which are proposed in [62], the $\mathbf{L}[\cdot]$ operator writes, in matrix form:

$$\mathbf{L}[\cdot] = \left\{ \mathbf{A}_i \frac{\mathrm{d}}{\mathrm{d}x_i} + \mathbf{B}_i \right\}, \quad 0 < x_i < e_i, \ \ i = 1, n, \tag{11}$$

where

$$\mathbf{A}_i = \begin{bmatrix} 0 & 1/\beta_i \\ \beta_i/\tau_i & 0 \end{bmatrix}, \quad \mathbf{B}_i = \begin{bmatrix} 0 & 0 \\ 0 & 1/\tau_i \end{bmatrix}.$$

The eigenvalue problem, for the $\mathbf{L}[\cdot]$ operator – we remind that only one eigenvalue problem has to be solved – is explicited by:

$$\mathbf{A}_i \frac{\mathrm{d}\mathbf{Z}_i}{\mathrm{d}x_i} + \mathrm{B}_i \boldsymbol{Z}_i = \omega \mathbf{Z}_i, \quad 0 < x_i < e_i, \ \ i = 1, n, \tag{12a}$$

$$\mathbf{Z}_{i+1}(0) = \mathbf{Y}_i \mathbf{Z}_i(e_i), \quad i = 1, n-1, \tag{12b}$$

where the thermal resistance matrix $\mathbf{Y}_i$ has the same form as in pure conduction context

$$\mathbf{Y}_i = \begin{bmatrix} 1 & -y_i \\ 0 & 1 \end{bmatrix}.$$

The external boundary conditions write

$$\mathbf{Z}_1(0) = \mathbf{Y}_1 \cdot \begin{pmatrix} 0 \\ q_1(0) \end{pmatrix}, \tag{12c}$$

$$\begin{pmatrix} 0 \\ q_n(e_n) \end{pmatrix} = \mathbf{Y}_{n+1} \cdot \mathbf{Z}_n(e_n). \tag{12d}$$

The analytical solution of Eq. (12a) is

$$\mathbf{Z}_i(x_i) = \mathrm{Exp}(x_i \mathbf{A}_i^{-1}(\omega \mathbf{I} - \mathbf{B}_i)) \mathbf{Z}_i(0), \tag{13}$$

which shows, by setting $x_i = e_i$, that the transfer matrix of the ith layer writes

$$\gamma_i(\omega, e_i) = \mathrm{Exp}(e_i \mathbf{A}_i^{-1}(\omega \mathbf{I} - \mathbf{B}_i)). \tag{14}$$

What follows is the same as in the pure conduction case [59–63]; by eliminating the interface unknowns, $\mathbf{Z}_i(0)$, thanks to Eq. (12b), the interface $i/i+1$ is connected to the interface $i-1/i$ through the matrix product $\mathbf{Y}_{i+1}\gamma_i(\omega, e_i)$, and, when all the interface unknowns are eliminated step by step we obtain, owing to Eqs. (12c) and (12d) the final result

$$\begin{bmatrix} 0 \\ q_n(e_n) \end{bmatrix} = \mathbf{Y}_{n+1} \cdot \gamma_n(\omega, e_n) \cdots \mathbf{Y}_{i+1} \cdot \gamma_i(\omega, e_i) \cdots \mathbf{Y}_2 \cdot \gamma_1(\omega, e_1) \cdot \mathbf{Y}_1 \cdot \begin{bmatrix} 0 \\ q_1(0) \end{bmatrix}. \tag{15}$$

The matrix product – in descending order – which appears in Eq. (15) is the overall transfer matrix of the multifilmed medium. The eigenvalue equation is then obtained by canceling the first off-diagonal coefficient of the overall transfer matrix so that the boundary conditions (12c) and (12d) are verified for a non-zero flux density. The corresponding equation is, as in pure conduction problems, the characteristic equation.

The methodology coincidence between the pure conduction case and the hyperbolic conduction case must not hide a fundamental difference: in the former case, the eigenelements are real elements, in the latter case they are complex elements. In this spirit, the transfer matrix of the ith layer may be explicited [1] as

$$\gamma_i(\omega, e_i) = \begin{bmatrix} \cos \mu_i e_i & -\dfrac{\mu_i}{\beta_i \omega} \sin \mu_i e_i \\ \dfrac{\beta_i \omega}{\mu_i} \sin \mu_i e_i & \cos \mu_i e_i \end{bmatrix}, \tag{16}$$

where we set

[1] The fastest method consists in eliminating the flux density, $q_i(x_i)$, between the two equations of system (12a) so as to obtain a differential second-order equation for $u_i(x_i)$, the first component of $\mathbf{Z}_i(x_i)$. Its general solution is trivial; $q_i(x_i)$, is then calculated thanks to the second equation of system (12a), the result (16) proceeds by identifying both components of $\mathbf{Z}_i(x_i)$ to the RHS of Eq. (13).

$\mu_i^2 = \omega(1 - \omega\tau_i)$.

But, although we defined the matrix coefficients with trigonometric functions – which strengthens the similarity to purely diffusive matrices [62] – it must be clear that since ω and μ_i are complex-valued numbers, sine and cosine functions are complex-valued functions. Nevertheless, for media with a small number of layers, analytical expressions of the overall transfer matrix will be known, the same will go for the eigenvalue equation.

4.2. Finite integral transform. Solution of 1-D problems

The direct transform writing (8) simplifies because of the prescribed direction of the flux density. With the used local reduced variables, it writes

$$\hat{p}_k(t) = \frac{1}{\langle \mathbf{Z}^k, \mathbf{Z}^{*k} \rangle} \sum_{i=1}^{n} \int_0^{e_i} \left(\beta_i \theta_i(x_i,t) \bar{v}_i^k + \frac{\tau_i}{\beta_i} \psi_i(x_i,t) \bar{r}_i^k \right) \mathrm{d}x_i. \tag{17}$$

By applying this transform to the homogeneous $\mathbf{T}^{\mathrm{H}}$ problem which is deduced from the full problem $\mathbf{T}$ by the translation – $\mathbf{T}^{\mathrm{S}}$, we obtain Eq. (9a) whose solution is still given by Eq. (10). Owing to the 1-D character of the problem, the quasi-stationary field is always analytically known so that the source-term $\hat{s}_k$ of the Eq. (9a) which writes now

$$\hat{s}_k = \frac{1}{\langle \mathbf{Z}^k, \mathbf{Z}^{*k} \rangle} \sum_{i=1}^{n} \int_0^{e_i} \left(\beta_i \frac{\partial T_i^{\mathrm{S}}}{\partial t} \bar{v}_i^k + \frac{\tau_i}{\beta_i} \frac{\partial \varphi_i^{\mathrm{S}}}{\partial t} \bar{r}_i^k \right) \mathrm{d}x_i \tag{18}$$

will always have an analytical expression. In the same way, if the initial condition is given analytically, the integral transform $\hat{p}_k^{\mathrm{H}}(0)$ could be analytically calculated and similarly for the solution $\hat{p}_k^{\mathrm{H}}(t)$ (Eq. (10)).

This (generally) analytical solution of the posed 1-D problem is thus

$$\mathbf{T}(x,t) = \left\{ \begin{array}{l} T_i(x_i,t) = T_i^{\mathrm{S}}(x_i,t) + \sum_k \hat{p}_k^{\mathrm{H}}(t) u_i^k(x_i) \\ \varphi_i(x_i,t) = \varphi_i^{\mathrm{S}}(x_i,t) + \sum_k \hat{p}_k^{\mathrm{H}}(t) q_i^k(x_i) \end{array}, \quad i = 1,n \right\}.$$

5. Conclusion

A constructive method of solution of hyperbolic conduction problems in heterogeneous media is described in this paper. It distinguishes from other finite integral transform techniques through the constant use of both thermal state variables, i.e. temperature and flux density, thanks to a formalism which retains throughout its development, the notion of coupled system of partial differential equations and the associated notion of solution vector-function. The finite integral transform which is based on a dot product which suits these vector-functions gives to the technique separability properties of time and space variables which have not been exploited until now. Thus, in the transformed space, not only is the (scalar) coefficient of a given rank in the expansion of the solution not coupled to the coefficients of other ranks, but also this coefficient is the same for the temperature as for the three components (3D problems) of the flux density. Both state variables are thus obtained through the same unitary formulation.

For 1D problems (multifilmed media) we showed that the notion of transfer matrix which is widely used in purely conduction problems generalizes very naturally to the hyperbolic conduction case. The analytical processing of hyperbolic 1D problems in non-homogeneous media is from then on conceivable.

The continuous use of state variables requires the knowledge of the initial temperature and flux density fields, instead of temperature and temperature time-derivative fields (for most standard situations) or flux density and flux density time-derivative (for special applications in 1D problems). But, not only is this requirement not a difficulty since all problems which are considered in the literature on the subject use initial conditions stemming from stationary regimes (where, of course, temperature gradients and therefore the flux density are known as soon as the temperature field is given) but also this requirement calls for less hypothesis than in standard hyperbolic equation applications; moreover it bypasses the mathematical difficulty of using generalized functions when studying the response to step-functions.

The application prospects of this work seem to be interesting from different points of view.

Firstly, the used approach, in which integral transforms are applied to vector-functions, might open a new way for studying transient problems which are governed by a set of linearly coupled partial differential equations.

Secondly, the applications of this approach to conduction problems in non-Fourier context might prove to be useful in comparing the results of macroscopic linear models (assuming the continuity hypothesis) to the results of microscopic or molecular models (not assuming the continuity hypothesis). Such results are already available in homogeneous media and even though this new method has been firstly built to allow the separation of variables in non-homogeneous media, it can also provide interesting results in homogeneous media. We propose, in a following paper to test, on this basis, the pertinence of the Cattaneo–Vernotte internal constitutive law.

Appendix A. Adjoint operator

The $\mathbf{L}[\cdot]$ and $\mathbf{L}^*[\cdot]$ operators are linked together by the basic relationship

$$\langle \mathbf{L}[\mathbf{Z}], \mathbf{Z}^* \rangle = \langle \mathbf{Z}, \mathbf{L}^*[\mathbf{Z}^*] \rangle, \tag{A.1}$$

where $\mathbf{Z}$ and $\mathbf{Z}^*$ are complex vector piecewise continuous and derivable fields.

The interface conditions (2a) and (2b) are verified by the $\mathbf{Z}$ field and the corresponding homogeneous boundary conditions are

$$U_\alpha - y_\alpha \vec{\mathbf{Q}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} = 0, \quad P \in C_\alpha. \tag{A.2}$$

Let $\mathbf{L}^*[\cdot]$ be the following operator:

$$\mathbf{L}^*[\mathbf{Z}^*] = \left\{ \begin{array}{l} -\dfrac{1}{\rho_i c_i} \vec{\nabla} \cdot \vec{\mathbf{R}}_i \\ -\dfrac{\lambda_i}{\tau_i} \vec{\nabla} V_i + \dfrac{\vec{\mathbf{R}}_i}{\tau_i} \end{array}, \quad i = 1, n \right\}, \tag{A.3}$$

where the $\mathbf{Z}^*$ field verifies the following interface and boundary conditions:

$$\vec{\mathbf{R}}_i \cdot \vec{\mathbf{n}}_{ij} + \vec{\mathbf{R}}_j \cdot \vec{\mathbf{n}}_{ji} = 0, \quad P \in C_{ij}, \tag{A.4a}$$

$$V_i - V_j = -y_{ij} \vec{\mathbf{R}}_i \cdot \vec{\mathbf{n}}_{ij}, \quad P \in C_{ij}, \tag{A.4b}$$

$$V_\alpha + y_\alpha \vec{\mathbf{R}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} = 0, \quad P \in C_\alpha. \tag{A.4c}$$

We notice that these conditions are deduced from the corresponding conditions of the problem by a change in the sign of the flux density.

The LHS of the Eq. (A.1) may be written

$$\langle \mathbf{L}[\mathbf{Z}], \mathbf{Z}^* \rangle = \sum_{i=1}^{n} \int_{D_i} \left[(\vec{\nabla} \cdot \vec{\mathbf{Q}}_i) \bar{V}_i + \frac{\vec{\mathbf{Q}}_i \cdot \vec{\bar{\mathbf{R}}}_i}{\lambda_i} + (\vec{\nabla} U_i) \cdot \vec{\bar{\mathbf{R}}}_i \right] \mathrm{d}v_i. \tag{A.5}$$

By using the identity

$$\vec{\nabla} \cdot (\bar{V}_i \vec{\mathbf{Q}}_i) = \bar{V}_i \vec{\nabla} \cdot \vec{\mathbf{Q}}_i + (\vec{\nabla} \bar{V}_i) \cdot \vec{\mathbf{Q}}_i$$

we make the divergence of the $\bar{V}_i \vec{\mathbf{Q}}_i$ vector, in the first term of the RHS of Eq. (A.5), appear. Then, thanks to the divergence-flux theorem, we transform the volume integral on each (D_i) sub-domain into a surface integral on each (C_i) boundary of the corresponding sub-domain. We obtain

$$\langle \mathbf{L}[\mathbf{Z}], \mathbf{Z}^* \rangle = \sum_{i=1}^{n} \int_{C_i} \bar{V}_i \vec{\mathbf{Q}}_i \cdot \vec{\mathbf{n}}_i \, \mathrm{d}s_i + \sum_{i=1}^{n} \int_{D_i} \left[\frac{\vec{\mathbf{Q}}_i \cdot \vec{\bar{\mathbf{R}}}_i}{\lambda_i} + (\vec{\nabla} U_i) \cdot \vec{\bar{\mathbf{R}}}_i - (\vec{\nabla} \bar{V}_i) \cdot \vec{\mathbf{Q}}_i \right] \mathrm{d}v_i. \tag{A.6}$$

The first arithmetic summation is split into two parts so as to distinguish the interfaces C_{ij} – which appear twice – from the external boundaries (C_α), which gives

$$\sum_{i=1}^{n} \int_{C_i} \bar{V}_i \vec{\mathbf{Q}}_i \cdot \vec{\mathbf{n}}_i \, \mathrm{d}s_i = \sum_{i \neq j} \int_{C_{ij}} (\bar{V}_i \vec{\mathbf{Q}}_i \cdot \vec{\mathbf{n}}_{ij} + \bar{V}_j \vec{\mathbf{Q}}_j \cdot \vec{\mathbf{n}}_{ji}) \, \mathrm{d}s_{ij} + \sum_{\alpha} \int_{C_\alpha} (\bar{V}_\alpha \vec{\mathbf{Q}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0}) \, \mathrm{d}s_\alpha.$$

By following the same calculation scheme as above, the RHS of Eq. (A.1) writes first

$$\langle \mathbf{Z}, \mathbf{L}^*[\mathbf{Z}^*] \rangle = \sum_{i=1}^{n} \int_{D_i} \left[-U_i \vec{\nabla} \cdot \vec{\bar{\mathbf{R}}}_i + \frac{\vec{\mathbf{Q}}_i \cdot \vec{\bar{\mathbf{R}}}_i}{\lambda_i} - \vec{\mathbf{Q}}_i \cdot \vec{\nabla} \bar{V}_i \right] \mathrm{d}v_i$$

and then

$$\langle \mathbf{Z}, \mathbf{L}[Z^*] \rangle = \sum_{i=1}^{n} \int_{C_i} -U_i \vec{\bar{\mathbf{R}}}_i \cdot \vec{\mathbf{n}}_i \, \mathrm{d}s_i + \sum_{i=1}^{n} \int_{D_i} \left[\frac{\vec{\mathbf{Q}}_i \cdot \vec{\bar{\mathbf{R}}}_i}{\lambda_i} + (\vec{\nabla} U_i) \cdot \vec{\bar{\mathbf{R}}}_i - (\vec{\nabla} \bar{V}_i) \cdot \vec{\mathbf{Q}}_i \right] \mathrm{d}v_i \tag{A.7}$$

with

$$\sum_{i=1}^{n} \int_{C_i} -U_i \vec{\bar{\mathbf{R}}}_i \cdot \vec{\mathbf{n}}_i \, \mathrm{d}s_i = -\sum_{i \neq j} \int_{C_{ij}} (U_i \vec{\bar{\mathbf{R}}}_i \cdot \vec{\mathbf{n}}_{ij} + U_j \vec{\bar{\mathbf{R}}}_j \cdot \vec{\mathbf{n}}_{ji}) \, \mathrm{d}s_{ij} - \sum_{\alpha} \int_{C_\alpha} U_\alpha \vec{\bar{\mathbf{R}}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} \, \mathrm{d}s_\alpha.$$

The volume integrals of Eqs. (A.6) and (A.7) are identical. It thus remains to show that the surface integrals are also identical so as to prove that the adjoint problem is effectively defined by Eqs. (A.3),(A.4a)–(A.4c).

The difference between the surface integrals writes

$$\sum_{i \neq j} \int_{C_{ij}} (\bar{V}_i \vec{\mathbf{Q}}_i \cdot \vec{\mathbf{n}}_{ij} + \bar{V}_j \vec{\mathbf{Q}}_j \cdot \vec{\mathbf{n}}_{ji} + U_i \vec{\bar{\mathbf{R}}}_i \cdot \vec{\mathbf{n}}_{ij} + U_j \vec{\bar{\mathbf{R}}}_j \cdot \vec{\mathbf{n}}_{ji}) \, \mathrm{d}s_{ij} + \sum_{\alpha} \int_{C_\alpha} (\bar{V}_\alpha \vec{\mathbf{Q}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} + U_\alpha \vec{\bar{\mathbf{R}}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0}) \, \mathrm{d}s_\alpha.$$

Owing to the boundary conditions Eqs. (A.2) and (A.4c) we obtain

$$\bar{V}_\alpha \vec{\mathbf{Q}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} + U_\alpha \vec{\bar{\mathbf{R}}}_\alpha \cdot \vec{\mathbf{n}}_{\alpha 0} = \bar{V}_\alpha \frac{U_\alpha}{y_\alpha} - U_\alpha \frac{\bar{V}_\alpha}{y_\alpha} = 0.$$

The boundary contribution vanishes. Owing to the interface conditions

$$\vec{\mathbf{Q}}_i \cdot \vec{\mathbf{n}}_{ij} = \frac{U_i - U_j}{y_{ij}}, \quad \vec{\mathbf{Q}}_j \cdot \vec{\mathbf{n}}_{ji} = \frac{U_j - U_i}{y_{ji}},$$

$$\vec{\bar{\mathbf{R}}}_i \cdot \vec{\mathbf{n}}_{ij} = \frac{\bar{V}_j - \bar{V}_i}{y_{ij}}, \quad \vec{\bar{\mathbf{R}}}_j \cdot \vec{\mathbf{n}}_{ji} = \frac{\bar{V}_i - \bar{V}_j}{y_{ji}},$$

the integrant, in the interface integrals writes

$$\bar{V}_i U_i\left(\frac{1}{y_{ij}}-\frac{1}{y_{ij}}\right)+\bar{V}_i U_j\left(-\frac{1}{y_{ij}}+\frac{1}{y_{ji}}\right)+\bar{V}_j U_j\left(\frac{1}{y_{ji}}-\frac{1}{y_{ji}}\right)+\bar{V}_j U_i\left(-\frac{1}{y_{ji}}+\frac{1}{y_{ij}}\right)$$

and, by noticing that $y_{ij}=y_{ji}$ is a consequence of the flux continuity on the interface, we observe that the contribution of interfaces vanishes too.

We conclude that the Eqs. (A.3),(A.4a)–(A.4c) define the adjoint problem.

Appendix B. Some properties of the eigenelements

Let $\mathbf{Z}^k$ and $\mathbf{Z}^{*l}$ be two solution vector fields of the eigenvalue problems

$$\mathbf{L}\left[\mathbf{Z}^k\right]=\omega_k\mathbf{Z}^k,\quad \mathbf{L}\left[\mathbf{Z}^{*l}\right]=\omega_l^*\mathbf{Z}^{*l},$$

where $\mathbf{Z}^k$ (resp. $\mathbf{Z}^{*l}$) verifies the interface conditions and the homogeneous boundary conditions of the problem (resp. of the adjoint problem).

In accordance with the basic relationship Eq. (A.1), we may write

$$\langle\mathbf{L}\left[\mathbf{Z}^k\right],\mathbf{Z}^{*l}\rangle=\langle\mathbf{Z}^k,\mathbf{L}^*\left[\mathbf{Z}^{*l}\right]\rangle,$$

and, since $\mathbf{Z}^k$ (resp. $\mathbf{Z}^{*l}$) are solutions of the above eigenvalue problems

$$\langle\omega_k\mathbf{Z}^k,\mathbf{Z}^{*l}\rangle=\langle\mathbf{Z}^k,\omega_l^*\mathbf{Z}^{*l}\rangle$$

or

$$(\omega_k-\bar{\omega}_l^*)\langle\mathbf{Z}^k,\mathbf{Z}^{*l}\rangle=0. \tag{B.1}$$

The $\mathbf{Z}^k$ field is orthogonal to the $\mathbf{Z}^{*l}$ field.

In an explicit form, the previous first eigenvalue problem, writes

$$\left\{\begin{array}{l}\dfrac{1}{\rho_i c_i}\vec{\nabla}\cdot\vec{\mathbf{Q}}_i^k=\omega_k U_i^k\\ \dfrac{\lambda_i}{\tau_i}\vec{\nabla}U_i^k+\dfrac{\vec{\mathbf{Q}}_i^k}{\tau_i}=\omega_k\vec{\mathbf{Q}}_i^k\end{array}\right\},\quad i=1,n$$

$$\vec{\mathbf{Q}}_i^k\cdot\vec{\mathbf{n}}_{ij}+\vec{\mathbf{Q}}_j^k\cdot\vec{\mathbf{n}}_{ji}=0,$$
$$U_i^k-U_j^k=y_{ij}\vec{\mathbf{Q}}_i^k\cdot\vec{\mathbf{n}}_{ij}$$
$$\forall i\neq j,\quad P\in C_{ij},$$
$$U_\alpha^k-y(P)\vec{\mathbf{Q}}_\alpha^k\cdot\vec{\mathbf{n}}_{\alpha 0}=0,\quad P\in C_\alpha.$$

Since the thermal properties of the media are real-valued coefficients, the conjugate problem may be written

$$\left\{\begin{array}{l}-\dfrac{1}{\rho_i c_i}\vec{\nabla}\cdot(-\vec{\mathbf{Q}}_i^k)=\bar{\omega}_k\bar{U}_i^k\\ -\dfrac{\lambda_i}{\tau_i}\vec{\nabla}\bar{U}_i^k-\dfrac{\vec{\mathbf{Q}}_i^k}{\tau_i}=\bar{\omega}_k(-\vec{\mathbf{Q}}_i^k)\end{array}\right\},\quad i=1,n$$

$$-\vec{\mathbf{Q}}_i^k\cdot\vec{\mathbf{n}}_{ij}-\vec{\mathbf{Q}}_j^k\cdot\vec{\mathbf{n}}_{ji}=0,$$
$$\bar{U}_i^k-\bar{U}_j^k=-y_{ij}(-\vec{\mathbf{Q}}_i^k\cdot\vec{\mathbf{n}}_{ij})$$
$$\forall i\neq j,\quad P\in C_{ij}$$
$$\bar{U}_\alpha^k+y(P)(-\vec{\mathbf{Q}}_\alpha\cdot\vec{\mathbf{n}}_{\alpha 0})=0,\quad P\in C_\alpha.$$

This writing shows, firstly that, if ω_k is an eigenvalue, $\bar{\omega}_k$ is also an eigenvalue, secondly that the vector field which is defined by

$$\left\{V_i^k=\bar{U}_i^k\quad\text{and}\quad\vec{\mathbf{R}}_i^k=-\vec{\bar{\mathbf{Q}}}_i^k,\quad i=1,n\right\}$$

is a solution of the eigenvalue problem

$$\mathbf{L}^*\left[\mathbf{Z}^{*k}\right]=\bar{\omega}_k\mathbf{Z}^{*k}$$

when $\mathbf{Z}^{*k}=(V^k,\vec{\mathbf{R}}^k)^t$ verifies the homogeneous boundary conditions of the adjoint problem.

We conclude that this vector field is a solution of the adjoint eigenvalue problem when $\omega_k^*=\bar{\omega}_k$. The orthogonality property Eq. (B.1) shows that the above-defined $\mathbf{Z}^{*k}$ field is the $\mathbf{Z}^k$ homologous field.

The eigenelements of the adjoint problem are immediately deduced from the eigenelements of the problem.

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